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ZG_n-Modules, G_n Cyclic of Square-Free Order n

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In the companion paper (*J. Algebra* 93 (1985), 1–116) all finitely generated modules over a class of rings called *Dedekind-like* are described, with emphasis on the behavior of these modules in direct sums. The present paper begins by showing that the integral group ring ZG_n is Dedekind-like. Some properties of ZG_n-modules are studied that do not hold for Dedekind-like rings in general. The modules studied do not necessarily have torsion-free abelian groups.

Our most dramatic example is that for every integer $m \geqslant 2$ there is a module M over some ZG_n such that

(0.1) M is the direct sum of s indecomposable ZG_n-modules for every s in the interval $2 \leqslant s \leqslant m$.

Since examples like this are no longer surprising for infinitely generated modules we remark that, in (0.1), $(M, +)$ is a free abelian group of finite rank. See Theorem 2.8.

Despite this example, ZG_n-modules are quite well behaved: The torsion-free rank of every indecomposable, finitely generated ZG_n-module is $\leqslant n$. (See Proposition 3.1.) Thus the order of quantifiers is important in setting up the example in (0.1): First m is given; then (with the help of Dirichlet's theorem on infinitely many primes in arithmetic progression) we find n and the ZG_n-module M.

To state some other ways in which ZG_n-modules (always finitely generated) are well behaved, let P be a maximal ideal of ZG_n. (i) Modules over the P-localized group ring $(ZG_n)_P$ satisfy the Krull–Schmidt theorem. (ii) If M is an indecomposable ZG_n-module, then M_P is the direct sum of at most two indecomposable $(ZG_n)_P$-modules. See Theorem 2.9. Both of these properties are in marked contrast to what happens if ZG_n is localized at a prime p of Z. See Remarks 2.10. We remark that (ii), but not (i), holds for all Dedekind-like rings.

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Local versus Global Isomorphism

One of the most interesting properties of Dedekind-like rings R is that, if M is any (finitely generated) R-module, the number of non-isomorphic R-modules N such that

$$(0.2) \qquad M_P \cong N_P \qquad \text{(for all maximal ideals } P \text{ of } R)$$

divides the order $|\operatorname{Pic} R|$ of the Picard group of R. If M has projective dimension 1 and is faithful, then the number of such N actually *equals* $|\operatorname{Pic} R|$. The ZG_n-modules of projective dimension 1 are not well known, because all ZG_n-lattices have projective dimension 0 or ∞. It is easy to see that the torsion subgroup of every non-artinian, *indecomposable* ZG_n-module must be n-torsion (i.e., every element is annihilated by some power if n) (Proposition 3.1). In Theorem 3.2 we show that, if X is any finite n-torsion group without direct summands of prime order, then the additive group $ZG_n \oplus X$ can be given the structure of an *indecomposable*, faithful ZG_n-module of projective dimension 1 in at least $|\operatorname{Pic} ZG_n|$ non-isomorphic ways.

Incidentally, the reason for the name "Dedekind-like" is that, as with Dedekind *domains* R, direct-sum behavior of all R-modules is controlled by that of projective modules of rank 1. The same is true of local versus global behavior of R-modules. The main difference is that, in our more general situation, we have to simultaneously consider projective modules of rank 1 over all rings between R and its integral closure (in its total quotient ring). See [L2].

Dedekind-like Rings; Fixed Notation

To define a Dedekind-like ring, let

$$(0.3) \qquad \tilde{R} = \bigoplus_c R_c \qquad \text{and} \qquad \bar{R} = \bigoplus_k \bar{R}_k$$

where each R_c is a Dedekind domain (but not a field) and each $\bar{R}_k$ is a field. The summations extend over unspecified, but fixed index sets.

Let f and $g : \tilde{R}$ onto $\bar{R}$ be ring homomorphisms that satisfy the *independence condition*

$$(0.4) \qquad K_f + K_g = \tilde{R} \qquad (K_f = \ker f,\ K_g = \ker g).$$

Then define the *Dedekind-like ring* R to be the (generalized) *pullback*

$$(0.5) \qquad R = \operatorname{pbk}(f, g) = \{x \in \tilde{R} \mid f(x) = g(x)\}.$$

For $R = ZG_n$ we have

$$(0.6) \qquad \tilde{R} = \bigoplus_{d \mid n} Z[\zeta_d]$$

where ζ_d is a primitive dth root of unity (in the complex numbers), and the direct sum extends over all divisors d of n, including 1 and n. Each $\bar{R}_k$ is a finite field whose characteristic divides n. For an actual description of these fields and the homomorphisms f and g, see Notations 1.1 and 2.1. This description generalizes the well-known fact that when $n = p$, a prime, ZG_p is isomorphic to the pullback of $Z \oplus Z[\zeta_p]$ determined by the diagram

$$(0.7)\qquad \begin{tikzcd} & Z \arrow[dr, "f"] & \\ ZG_p \arrow[ur] \arrow[dr] & & Z/pZ \\ & Z[\zeta_p] \arrow[ur, "g"'] & \end{tikzcd} \qquad \text{where} \qquad \begin{tikzcd} & 1 \arrow[dr] & \\ x_p \arrow[ur] \arrow[dr] & & 1 \\ & \zeta_p \arrow[ur] & \end{tikzcd}$$

that is, $ZG_n \cong \{(u, v) \in Z \oplus Z[\zeta_p] \mid f(u) = g(v)\}$.

With a small amount of additional work, we show that $Z[\zeta_q]\, G_n$ is Dedekind-like whenever ζ_q is a primitive qth root of unity and qn is square-free. See 1.6–1.8.

1. ZG_n and $Z[\zeta_q]\, G_n$ As Pullbacks

1.1. *Notation.* Let $G_n = \langle x_n \rangle$ denote a cyclic group of square-free order n, generated by an element x_n. In order to display ZG_n as a Dedekind-like ring, we fix the following notation.

Let $\tilde{R}$ be as in (0.6). For each d and p such that dp divides n and p is prime, let $(\mathscr{P}_{d,dp})$ be the diagram

$$(\mathscr{P}_{d,dp})\qquad \begin{tikzcd} Z[\zeta_d] \arrow[dr, "f_{d,dp}"] & \\ & Z[\zeta_d]/\langle p \rangle \\ Z[\zeta_{dp}] \arrow[ur, "g_{d,dp}"'] & \end{tikzcd} \qquad \text{where} \qquad \begin{tikzcd} \zeta_d \arrow[dr] & \\ & \zeta_d + \langle p \rangle. \\ \zeta_{dp} \arrow[ur] & \end{tikzcd}$$

Here $\langle p \rangle$ denotes the ideal generated by p. The proof that $g_{d,dp}$ is a well-defined ring homomorphism is contained in the proof of Theorem 1.2. Let

$$(1)\qquad \bar{R} = \bigoplus_{d,p} Z[\zeta_d]/\langle p \rangle \qquad \text{where } dp \mid n \text{ and } p \text{ is prime.}$$

Since n is square-free, p does not divide d; so $Z[\zeta_d]/\langle p\rangle$ is a direct sum of fields, hence $\bar{R}$ is a direct sum of fields [J, p. 44].

Define two ring homomorphisms f and g: $\tilde{R}$ onto $\bar{R}$ by using the maps $f_{d,dp}$ and $g_{d,dp}$ as coordinate homomorphisms. Then let R be the *pullback* of $\tilde{R}$ determined by diagrams $\mathscr{P}_{d,dp}$:

$$R = \operatorname{pbk}(f, g) = \{r \in \tilde{R} \mid f(r) = g(r)\}. \tag{2}$$

More explicitly, R is the set of elements $\{r_d\}_{d|n} \in \tilde{R}$ such that

$$f_{d,dp}(r_d) = g_{d,dp}(r_{dp}) \qquad \text{whenever } dp \mid n \text{ and } p \text{ is prime.} \tag{3}$$

1.2. THEOREM. (When n is square-free.) *$ZG_n \cong R = \operatorname{pbk}(f, g)$ via the ring isomorpphism given by $x_n \to \{\zeta_d\}_{d|n}$.*

The proof follows two lemmas.

1.3. LEMMA (Passman). *Let $K_1, \ldots, K_n$ be ideals of a commutative ring R, and suppose*

$$K_a + K_b + K_c = R \qquad \textit{whenever} \quad a < b < c. \tag{1}$$

Let $r_1, \ldots, r_n$ be elements of R such that, for all a and b,

$$r_a \equiv r_b \qquad (\operatorname{mod} K_a + K_b). \tag{2}$$

Then $\exists r \in R$ such that $r \equiv r_a$ $(\operatorname{mod} K_a)$ for all a.

Proof. If $n = 2$ we have $r_1 - r_2 = k_1 + k_2$ so the element we want is $r = r_1 - k_1 = r_2 + k_2$. Suppose now that $n > 2$. By induction on n we can suppose that there exists $s \in R$ such that $s \equiv r_a$ $(\operatorname{mod} K_a)$ whenever $a \neq n$. Then, for $a \neq n$ we have $s \equiv r_n$ $(\operatorname{mod} K_a + K_n)$ so

$$s - r_n \in \bigcap_{a=1}^{n-1} (K_a + K_n). \tag{3}$$

Let $M_a = K_a + K_n$ $(a \neq n)$. Then $M_a + M_b = R$ whenever $a \neq b$, by (1), hence $M_a \cap M_b = M_a M_b$. Therefore

$$\bigcap_{a=1}^{n-1} (K_a + K_n) = \prod_{a=1}^{n-1} (K_a + K_n) \subseteq \left(\bigcap_{a=1}^{n-1} K_a\right) + K_n.$$

Therefore, by (3), $s - r_n = i + k$ with $i \in \bigcap_{a=1}^{n-1} K_a$ and $k \in K_n$. The element we want is therefore

$$r = s - i = r_n + k. \quad \blacksquare$$

1.4. LEMMA. *Let a and b be distinct square-free positive integers, and Φ_a the cyclotomic polynomial of order a. Then in the polynomial ring $Z[x]$ we have*

$$
\begin{aligned}
\langle \Phi_a \rangle + \langle \Phi_b \rangle &= \langle \Phi_a \rangle + \langle p \rangle && \text{if} \quad b/a = p = \text{prime} \\
&= \langle \Phi_b \rangle + \langle p \rangle && \text{if} \quad a/b = p = \text{prime} \qquad (1) \\
&= Z[x] && \text{otherwise.}
\end{aligned}
$$

Proof. Combine Theorem 25.26 and isomorphism 25.28 in [CR]. ∎

1.5. *Proof of Theorem* 1.2. Define a ring homomorphism σ of the polynomial ring $Z[x]$ into $\tilde{R} = \bigoplus_{d \mid n} Z[\zeta_d]$ by $x \to \{\zeta_d\}$. Since $\ker \sigma = \langle x^n - 1 \rangle$, $\operatorname{im} \sigma \cong ZG_n$ via $x \to x_n$. So it suffices to prove that $\operatorname{im} \sigma$ is the ring $R = \operatorname{pbk}(f, g)$ described in Notation 1.1(2) and (3).

For each pair of divisors $a < b$ of n, let $\bar{R}(a, b)$ be the ring shown in $(1)_{a,b}$, and let $f_{a,b}$ and $g_{a,b}$ be the unique ring homomorphisms that make the two triangles in $(1)_{a,b}$ commute:

$$
(1)_{a,b} \qquad
\begin{tikzcd}
& Z[\zeta_a] \arrow[dr, "f_{a,b}"] & \\
Z[x] \arrow[ur, "\sigma_a"] \arrow[dr, "\sigma_b"'] \arrow[rr, two heads, "\text{nat hom}"] & & Z[x]/(\ker \sigma_a + \ker \sigma_b) = \bar{R}(a, b). \\
& Z[\zeta_b] \arrow[ur, "g_{a,b}"'] &
\end{tikzcd}
$$

By Lemma 1.4 we have

(2) $\ker \sigma_a + \ker \sigma_b = Z[x]$ whenever $a < b$ and $b/a \neq$ prime number.

From this we conclude

(3) $\ker \sigma_a + \ker \sigma_b + \ker \sigma_c = Z[x]$ whenever $a < b < c$.

We wish to conclude that $\sigma(Z[x])$ is the pullback $\operatorname{pbk}(f, g)$ of $\tilde{R}$ determined by the pairs of homomorphisms $f_{a,b}$, $g_{a,b}$; that is, $\sigma(Z[x])$ is the set of elements $\{r_d\}$ of $\tilde{R}$ such that

(4) $f_{a,b}(r_{a,b}) = g_{a,b}(r_b)$ for all pairs $a < b$ of divisors of n.

Commutativity of diagrams $(1)_{a,b}$ yields $(Z[x]) \subseteq \operatorname{pbk}(f, g)$. For the opposite inclusion, let $\{r_d\}$ be any element of R that satisfies (4), and take any pre-image $f_d \in Z[x]$ of each r_d, that is, $f_d(\zeta_d) = r_d$. Commutativity of $(1)_{a,b}$ shows $f_a \equiv f_b \pmod{\ker \sigma_a + \ker \sigma_b}$ for all a and b. This, together with (3) and Passman's lemma, yields $f \in Z[x]$ such that $f \equiv f_d \pmod{\ker \sigma_d}$ for all d; that is, $\sigma(f) = \{r_d\}$ as desired.

All that now remains is to show that pbk(f, g) is the subring of $\tilde{R}$ described in Notation 1.1(3). We consider two possibilities for a, b.

Suppose first that b/a is not a prime number. Then, by (2), $\bar{R}(a, b) = 0$, so we can ignore $(1)_{a,b}$ in the description of im σ.

Suppose next that $b/a = p =$ prime. By Lemma 1.4 we have

$$\text{(5)} \qquad \bar{R}(a, pa) \cong Z[x]/(\langle \Phi_a \rangle + \langle p \rangle) \cong Z[\zeta_a]/\langle p \rangle.$$

In $(1)_{a,pa}$ replace $\bar{R}(a, pa)$ by $Z[\zeta_a]/\langle p \rangle$, and replace "nat hom" by the map $x \to \zeta_a + \langle p \rangle$. Then $f_{a,pa}$ and $g_{a,pa}$ become the maps $\zeta_a \to \zeta_a + \langle p \rangle$ and $\zeta_{pa} \to \zeta_a + \langle p \rangle$, and this completes the proof. ∎

1.6. *Notation.* We now set up notation to describe the pullback structure of the group ring $Z[\zeta_q]\, G_n$, where qn is a square-free integer. Let

$$\text{(1)} \qquad \tilde{R} = \bigoplus_{d \mid n} Z[\zeta_{qd}].$$

For each d, p where dp divides n and p is prime, let $\mathscr{P}_{qd,qdp}$ be the diagram

$$(\mathscr{P}_{qd,qdp}) \quad \begin{tikzcd} Z[\zeta_{qd}] \arrow[dr, "f_{qd,qdp}"] & \\ & Z[\zeta_{qd}]/\langle p \rangle \\ Z[\zeta_{qdp}] \arrow[ur, "g_{qd,qdp}"'] & \end{tikzcd} \quad \text{where} \quad \begin{tikzcd} \zeta_{qd} \arrow[dr] & \\ & \zeta_{qd} + \langle p \rangle. \\ \zeta_{qdp} \arrow[ur] & \end{tikzcd}$$

Then let pbk(f, g) be the set of all elements $\tilde{r} = \{r_{qd}\}$ of R such that

$$\text{(2)} \qquad f_{qd,qdp}(r_{qd}) = g_{qd,qdp}(r_{qdp}) \qquad \text{whenever } dp \mid n \text{ and } p \text{ is prime}$$

or, more compactly, $f(\tilde{r}) = g(\tilde{r})$. Thus, in this generalization of the situation studied in Notation 1.1 we regard f and g has homomorphisms of $\tilde{R}$ onto $\bar{R}$, where

$$\text{(3)} \qquad \bar{R} = \bigoplus_{d,p} Z[\zeta_{qd}]/\langle p \rangle \qquad \text{where } dp \mid n \text{ and } p \text{ is prime.}$$

Since qn is square-free, we see that, in (3),

(4) $\quad p$ does not divide qd, hence $\bar{R}$ is a direct sum of fields.

1.7. Corollary. (Keep the notation of 1.6.) *The map*

$$(1)\qquad \pi\colon Z[\zeta_q]\,G_n \to \tilde{R} \qquad \text{given by} \quad \zeta_q x_n \to \{\zeta_{qd}\}_{d \mid n}$$

is an isomorphism of $Z[\zeta_q]\,G_n$ *onto* $\mathrm{pbk}(f, g)$.

Proof. Note that

$$(2)\qquad Z[\zeta_q]\,G_n \cong ZG_n \otimes_Z Z[\zeta_q].$$

In the notation of 1.1 (rather than 1.6) we have $ZG_n = \mathrm{pbk}(f, g)$. Thus it suffices to show that for any flat Z-module F we have

$$(3)\quad \mathrm{pbk}(f, g) \otimes_Z F = \mathrm{pbk}(f \otimes 1, g \otimes 1) \text{ where } \begin{cases} f \otimes 1\colon \tilde{R} \otimes_Z F \to \bar{R} \otimes_Z F \\ g \otimes 1\colon \tilde{R} \otimes_Z F \to \bar{R} \otimes_Z F. \end{cases}$$

The proof of [L2, Lemma 6.1]—with localization replaced by tensoring with an arbitrary flat module F—accomplishes this. ∎

1.8. Corollary. *The group rings* ZG_n *and more generally,* $Z[\zeta_q]\,G_n$ *with* qn (*hence* n) *square-free, are Dedekind-like.*

Proof. By Corollary 1.7 the group ring $Z[\zeta_q]\,G_n$ is isomorphic to the ring $R = \mathrm{pbk}(f, g)$ where, in the notation of 1.6, f and g are ring homomorphisms of $\tilde{R}$ onto $\bar{R}$. Thus we want to show that R is Dedekind-like.

A standard fact about cyclotomic fields is that every coordinate ring $Z[\zeta_{qd}]$ of $\tilde{R}$ is a Dedekind domain; and we have already observed in Notation 1.6(4) that $\bar{R}$ is a direct sum of fields.

Finally we verify Independence Condition (0.4). Each map $f_{qd,qdp}$ in diagram $\mathscr{P}_{qd,qdp}$ (see 1.6) can be extended to a homomorphism: $\tilde{R} \to \bar{R}$ by defining it to zero on each coordinate ring of $\tilde{R}$ other than $Z[\zeta_{qd}]$. Similarly, extend each $g_{qd,qdp}$ to a homomorphism: $\tilde{R} \to \bar{R}$. Then

$$(1)\qquad K_f = \bigcap_{d,p} \ker f_{qd,qdp} \qquad \text{and} \qquad K_g = \bigcap_{d,p} \ker g_{qd,qdp}.$$

To verify that $K_f + K_g = \tilde{R}$ it suffices to show that no maximal ideal M of $\tilde{R}$ contains both K_f and K_g.

Since M is maximal, it has a *principal coordinate* R_e, that is, M is the direct sum of its projection in R_e with $\oplus_{c \neq e} R_c$. Since M contains K_f and maximal ideals are prime, (1) shows that M contains some $\ker f_{qd,qdp}$. This kernel has a principal coordinate, R_{qd}. So $qd = e$ and M contains the prime number p. Note that p does not divide d.

Similarly, M contains the principal coordinate qcr of some $\ker g_{qc,qcr}$. So

$qcr = e = qd$ and M contains the prime number r. Since r divides $cr = d$, we see that M contains two distinct prime numbers p and r. This yields the contradiction that $1 \in M$ and completes the proof. ∎

2. Number of Indecomposables

2.1. *Notation.* Let $R = \mathrm{pbk}(f, g)$ be a Dedekind-like ring as in (0.3)–(0.5). In this section and the next it is necessary to view f and g as collections of maps f_k and g_k from the coordinate rings R_c of $\tilde{R}$ to the coordinate fields $\bar{R}_k$ of $\bar{R}$.

For each k there is a diagram

$$(\mathscr{P}_k) \qquad \begin{tikzcd} R_{i(k)} \arrow[dr, "f_k"] & \\ & \bar{R}_k \\ R_{j(k)} \arrow[ur, "g_k"'] & \end{tikzcd} \qquad [\text{possibly } i(k) = j(k)]$$

such that R is the set of tuples $\{r_c\}$ in $\tilde{R} = \bigoplus_c R_c$ such that

$$(1) \qquad f_k(r_{i(k)}) = g_k(r_{j(k)}) \qquad (\forall\, k).$$

This notation $i(k), j(k), f_k, g_k$ will remain fixed throughout this paper.

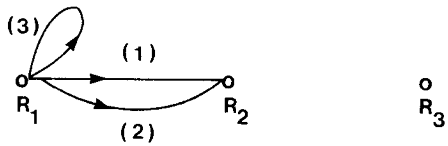
To construct diagram $\mathscr{P}_k$ let φ_k be the composition of $f\colon \tilde{R} \to \bar{R}$ with coordinate projection: $\bar{R} \to \bar{R}_k$. Since $\ker \varphi_k$ is a maximal ideal of $\tilde{R}$, it is direct sum of maximal ideal of some $R_{i(k)}$ with all of the other coordinate rings R_c of $\tilde{R}$. Let f_k be the restriction of f to $R_{i(k)}$. Define $R_{j(k)}$ and g_k similarly, using g instead of f. Then it is straightforward to check that R is the set of elements $\{r_c\}$ of $\tilde{R}$ such that (1) holds. See [L2, Sect. 3] for more discussion of this.

2.2. *Graph of R.* To visualize the system of pullback homomorphisms f_k and g_k we introduce the directed graph (actually, multigraph, since more than one edge is allowed to connect a pair of points) $\mathrm{gph}\, R$.

The vertices of $\mathrm{gph}\, R$ consist of one point, labeled by R_c, for each coordinate ring R_c of $\tilde{R}$. For each coordinate ring $\bar{R}_k$ of $\bar{R}$, $\mathrm{gph}\, R$ contains what we call a *k-pullback edge*, beginning at point $R_{i(k)}$, ending at point $R_{j(k)}$, and labeled (k). This k-pullback edge indicates that coordinate rings $R_{i(k)}$ and $R_{j(k)}$ are "connected via $\bar{R}_k$" in pullback diagram $\mathscr{P}_k$ of Notation 2.1.

Diagram (1) shows a specific, simple example of what gph R might look like:

(1)



Here each of $\tilde{R}$ and $\bar{R}$ has three coordinate rings. As we will see, a loop connecting a point to itself—like the 3-pullback edge in diagram (1)—never occurs for $R = ZG_n$ or $Z[\zeta_q]\, G_n$, although it can occur for general Dedekind-like rings. See [L2, 11.1]. We note:

(2) $\quad R$ is an indecomposable ring $\Leftrightarrow$ gph R is a connected graph.

This follows immediately from the fact that R is indecomposable if and only its only idempotent elements are 0 and 1. (Note that every coordinate of an idempotent element must equal 0 or 1.)

2.3. *Module* $S(\mathbf{e})$. Let $\mathbf{e}$ be an idempotent element of $\bar{R} = \bigoplus_k \bar{R}_k$. Then for each k, the kth coordinate e_k of $\mathbf{e}$ equals 0 or 1 in the field $\bar{R}_k$.

Gph $\mathbf{e}$ denotes the graph obtained from gph R by deleting the k-pullback edge whenever $e_k = 0$. Thus,

(3)
$$\text{gph } \mathbf{0} \text{ consists of vertices and no edges.}$$
$$\text{gph } \mathbf{1} = \text{gph } R.$$

Finally, define $S(\mathbf{e})$ to be the set of tuples $\{r_c\}$ in $\tilde{R} = \bigoplus_c R_c$ such that

(4)
$$f_k(r_{i(k)}) = g_k(r_{j(k)}) \qquad \text{whenever} \quad e_k = 1.$$

Thus the elements of $S(\mathbf{e})$ satisfy one pullback condition for each edge of gph $\mathbf{e}$. Observe that

(5)
$$S(\mathbf{0}) = \tilde{R} \qquad \text{and} \qquad S(\mathbf{1}) = R.$$

More generally, every $S(\mathbf{e})$ is a Dedekind-like ring between R and $\tilde{R}$. It is easy to see that $S(\mathbf{d}) \cdot S(\mathbf{e}) \subseteq S(\mathbf{de})$. Taking $\mathbf{d} = \mathbf{1}$ we see that $S(\mathbf{e})$ is an R-module.

LEMMA 2.4. *Let* $\mathbf{d}$ *and* $\mathbf{e}$ *be orthogonal idempotents of* $\bar{R}$ *such that* $\mathbf{d} + \mathbf{e} = \mathbf{1}$. *Then*

(i) $S(\mathbf{d}) \oplus S(\mathbf{e}) \cong R \oplus \tilde{R}$ *as* R-*modules.*

(ii) *If* gph $\mathbf{e}$ *has* s *connected components then* $S(\mathbf{e})$ *can be written as the direct sum of* s *indecomposable* R-*modules.*

Proof. (i) The proof given here uses results from [L2]. A longer but more elementary proof can be given following the lines of [L1, Lemma 3.1]. Main Theorem [L2, 10.2] associates, with each finitely generated R-module M (R Dedekind-like) an isomorphism class $\mathrm{cl}\, M$ of fractional R-ideals in such a way that

$$(1) \qquad M \cong N \Leftrightarrow \begin{cases} M_P \cong N_P & (\forall \text{ maximal ideals of } P \text{ of } R), \text{ and} \\ \mathrm{cl}\, M = \mathrm{cl}\, N. \end{cases}$$

Moreover

$$(2) \qquad \mathrm{cl}(M \oplus N) = (\mathrm{cl}\, M)(\mathrm{cl}\, N).$$

When M is isomorphic to a fractional R-ideal—that is, a finitely generated R-submodule of the total quotient ring $Q(R)$ of R that contains a unit of $Q(R)$, for example, $M = S(\mathbf{e})$—then $\mathrm{cl}\, M$ is the isomorphism class of M.

To compute the localizations needed in (1) we note that for each k there is a ring homomorphism of R onto $\bar{R}_k$, given by $f\colon R \to \bar{R}_k$. (Note that using g instead of f produces the same map: $R \to \bar{R}_k$ because $f = g$ on R.) Since $\bar{R}_k$ is a field, $\ker(R \to \bar{R}_k)$ is a maximal ideal of R. We have

$$(3) \qquad \begin{aligned} S(\mathbf{e})_P &\cong R_P && \text{if } P = \ker(R \to \bar{R}_k) \text{ and } e_k = 1 \\ &\cong \tilde{R}_P && \text{if } P = \ker(R \to \bar{R}_k) \text{ and } e_k = 0 \\ &\cong R_P && \text{if } P \text{ is any other maximal ideal of } R \end{aligned}$$

See [L2, 9.6], noting that $S(\mathbf{e}) = \mathrm{pbk}(\mathbf{e}; \tilde{R})$.

To compute the localizations in (i), choose k and let $P = \ker(R \to \bar{R}_k)$. Since $\mathbf{d}$ and $\mathbf{e}$ are orthogonal idempotents whose sum is $\mathbf{1}$, and since $\bar{R}_k$ is a field, one of d_k and e_k equals 0 and the other equals 1. Thus, after localization, the left-hand side of (i) becomes $\cong R_P \oplus \tilde{R}_P$ as desired.

If P is any other maximal ideal of R, then both sides of (i) are $\cong R_P \oplus R_P$. [Recall: $\tilde{R} = S(\mathbf{0})$.]

Finally we check ideal classes in (i). We ask: $S(\mathbf{d}) \cdot S(\mathbf{e}) \cong R \cdot \tilde{R}$? In fact, by [L2, 9.8] we have

$$S(\mathbf{d}) \cdot S(\mathbf{e}) = S(\mathbf{de}) = S(\mathbf{0}) = \tilde{R}$$

completing the proof of (i).

(ii) It suffices to show that $S(\mathbf{e})$, as a *ring*, is the direct sum of s indecomposable Dedekind-like rings. By working on one connected component at a time, we can suppose $\mathrm{gph}\, \mathbf{e}$ is connected, in which case $S(\mathbf{e})$ is indecomposable by 2.2(2). ∎

2.5. THEOREM. *Suppose, for a Dedekind-like ring R, that $\tilde{R}$ has m coordinate rings R_c and $\bar{R}$ has idempotents* $\mathbf{d}$ *and* $\mathbf{e}$ *such that*

(i) $\mathbf{d}$ *and* $\mathbf{e}$ *are orthogonal,* $\mathbf{d}+\mathbf{e}=\mathbf{1}$, *and* gph $\mathbf{d}$ *and* gph $\mathbf{e}$ *are both connected.*

Then $R \oplus \tilde{R}$ is the direct sum of s indecomposable R-modules for every s in the interval $2 \leqslant s \leqslant m+1$.

Proof. Connectedness of gph $\mathbf{d}$ and gph $\mathbf{e}$ shows that $S(\mathbf{d})$ and $S(\mathbf{e})$ are indecomposable, by Lemma 2.4(ii). So, by Lemma 2.4(i), the decomposition needed for $s=2$ is

$$S(\mathbf{d}) \oplus S(\mathbf{e}) \cong R \oplus \tilde{R}. \tag{1}$$

Now choose any edge of gph $\mathbf{e}$; say, its k-pullback edge. Then $e_k=1$ and $d_k=0$. Alter $\mathbf{d}$ and $\mathbf{e}$ by setting $d_k=1$ and $e_k=0$. Thus we have moved the k-pullback edge from gph $\mathbf{e}$ to gph $\mathbf{d}$.

Note. If an edge is removed from a connected graph G, then either G remains connected or else G becomes the union of two connected components. Thus gph $\mathbf{e}$ is now the union of one or two connected components, so $S(\mathbf{e})$ is either indecomposable or the direct sum of two indecomposable R-modules. $S(\mathbf{d})$ remains indecomposable because adding an edge to a connected graph leaves it connected.

Since (1) still holds, $R \oplus \tilde{R}$ is now expressed as the direct sum of 2 or 3 indecomposable modules.

Repeat the above procedure, moving the edges of gph $\mathbf{e}$ to gph $\mathbf{d}$ one at a time. Each time, we either leave the number of indecomposable summands unchanged, or increase it by 1. After all of the edges have been moved to gph $\mathbf{d}$, the left-hand side of (1) becomes

$$S(\mathbf{1}) \oplus S(\mathbf{0}) \qquad \text{which equals } R \oplus R_1 \oplus R_2 \oplus \cdots \oplus R_m.$$

This is the direct sum of $m+1$ indecomposable modules, as desired. ∎

2.6. *Remarks.* (gph R *when* $R=ZG_n$.) For $R=ZG_n$ (n square-free) we have

$$\tilde{R} = \bigoplus_{d \mid n} R_d \qquad \text{where} \quad R_d = Z[\zeta_d]. \tag{1}$$

Also, $\bar{R} = \bigoplus_{d,p} R_d/\langle p\rangle$ as shown in Notation 1.1(1).

In order to display $\bar{R}$ as a direct sum of fields, fix d and p. Since p does not divide d, the ideal $\langle p\rangle$ of R_d has a factorization $\langle p\rangle = P_1P_2\cdots P_g$ into a product of *distinct* maximal ideals of R_d. For later us we state the value of $g=g(d,p)$. (See [J, p. 44].)

(2) Let f be the smallest positive integer such that $p^f \equiv 1 \pmod{d}$. Then $g = g(d, p) = \varphi(d)/f$ where φ is the Euler totient function.

Thus $R_d/\langle p\rangle = \bigoplus_{h=1}^{g(d,p)} R_d/P_h$ and therefore,

$$\bar{R} = \bigoplus_{d,p,h} R_d/P_h \tag{3}$$

where P_h actually depends on d, p, and h. Let $\mathscr{P}_{d,dp,h}$ be the diagram

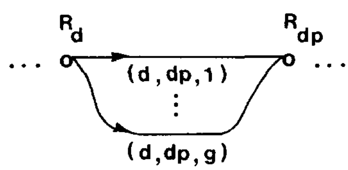
$$(\mathscr{P}_{d,dp,h}) \qquad \begin{tikzcd} R_d \arrow[dr, "f_{d,dp,h}"] & \\ & R_d/P_h \\ R_{dp} \arrow[ur, "g_{d,dp,h}"'] & \end{tikzcd} \qquad \text{where} \qquad \begin{tikzcd} \zeta_d \arrow[dr] & \\ & \zeta_d + P_h \\ \zeta_{dp} \arrow[ur] & \end{tikzcd}$$

and let $R = \mathrm{pbk}(f, g)$, that is, let R be the set of elements $\{r_d\}$ in $\tilde{R}$ such that, for every d, p, h,

$$f_{d,dp,h}(r_d) = g_{d,dp,h}(r_{dp}). \tag{4}$$

Then $R = \mathrm{pbk}(f, g)$ *is the same subring of* $\tilde{R}$ *as the ring* $R = \mathrm{pbk}(f, g)$ *defined in Notation* 1.1(3), hence $R \cong ZG_n$. We usually consider this isomorphism an identification, from now on, and write $R = ZG_n$.

Gph R *when* $R = ZG_n$. Its vertices consist of one point, labeled by R_d, for each divisor d of n. Its edges consist of one (d, dp, h)-pullback edge, beginning at R_d and ending at R_{dp}, for each d and p, p prime, such that $dp \mid n$, and each h:

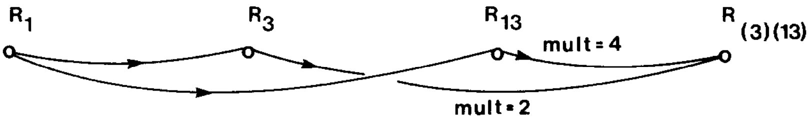
(5) 

The number g of edges joining R_d to R_{dp} in (5) is the number $g = g(d, p)$ in (2). *It is always* $\geqslant 1$.

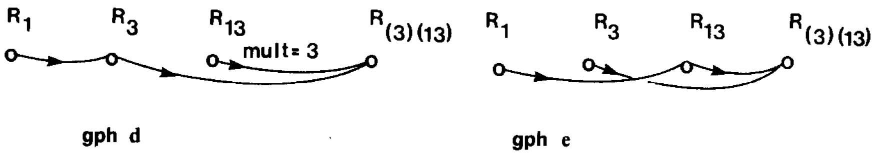
In the proof of the next proposition we draw the graph of ZG_{39}.

2.7. PROPOSITION. *Let* $R = ZG_{39}$. *Then* $\bar{R}$ *has idempotents* $\mathbf{d}$ *and* $\mathbf{e}$ *that satisfy condition* (i) *of Theorem* 2.5.

Proof. Since $39 = (3)(13)$, $\tilde{R} = R_1 \oplus R_3 \oplus R_{13} \oplus R_{39}$. Gph R is

(1) 

"Mult = 2" means that two edges connect R_3 to R_{39}, and these should be labeled (3, 39, 1) and (3, 39, 2) [but no edge labels are actually shown]. By assigning each edge in gph R to one of the graphs in (2),



(2)

we get two orthogonal idempotents **d** and **e** of $\bar{R}$ with connected graphs, and such that $\mathbf{d} + \mathbf{e} = 1$. For example, the (1, 3, 1)-coordinates of **d** and **e** are 1 and 0, respectively. ∎

2.8. THEOREM. *Let $s_0 \geqslant 2$. Then for some $R = ZG_n$ (n square-free) the R-module $R \oplus \tilde{R}$ can be written as the direct sum of s indecomposable R-modules for every s in the interval $2 \leqslant s \leqslant s_0$.*

Proof. It suffices to produce an infinite sequence of numbers $n(1) < n(2) < \cdots$ such that each $ZG_{n(i)}$ has more coordinate rings than the previous one and satisfies condition (i) of Theorem 2.5.

Let $n = p_1 p_2 \cdots p_t$, with each p_i prime and n square-free. The number of coordinate rings R_d of ZG_n equals the number of divisors of n. Since n is square-free this equals the number of subsets of $\{p_1, p_2, \ldots, p_t\}$, that is, 2^t. Hence ZG_{np} always has twice as many coordinate rings as ZG_n.

We already know that ZG_{39} has 4 coordinate rings and satisfies condition (i) of Theorem 2.5. So it suffices to prove the following inductive statement.

(1) Suppose $R = ZG_n$, with n odd and >1, satisfies 2.5(i). Then for some prime $p > n$, $R' = ZG_{np}$ also satisfies hypothesis 2.5(i).

By Dirichlet's theorem on primes in arithmetic progression there exists a prime $p > n$ such that $p \equiv 1 \pmod{n}$. Fix such a p. Then $p \equiv 1 \pmod{d}$ for every divisor d of n.

Separate the points of gph R' into two disjoint subsets: *Old* points, those that belong to gph R, and *new* points, those that do not. Thus *the point labeled by R_d is a new point if and only if $p \mid d$.* This is illustrated in chart (2).

Let **d** and **e** be the idempotents that work for $R = ZG_n$. Their graphs use only old points, as shown in chart (2):

$R_d \quad (d\,|\,n)$ $\cdots \ \circ \ \cdots$	$R_p \quad R_{dp}$ $\circ \ \cdots \ \circ \ \cdots$
(2) Old points ($\in$ gph R)	New points ($\notin$ gph R)
$\left.\begin{array}{l}\text{gph } \mathbf{d} \\ \text{gph } \mathbf{e}\end{array}\right\}$ (connected)	

We claim:

(3) Let R_d be an old point. Then, in gph R', point R_d is connected to point R_{dp} by *more than one* edge, if $d \neq 1$.

The number f in Remark 2.6(2) equals 1 since $p \equiv 1 \pmod d$. Therefore the number of edges connecting R_d to R_{dp} is $g = \varphi(d)/1$ which is >1 if $d \neq 1$, as claimed. (Note. $d \neq 2$ since $d\,|\,n$ and n is odd.)

To prove (1) we want to construct two connected graphs, gph $\mathbf{d}'$ and gph $\mathbf{e}'$, by assigning each edge of gph R' to exactly one of them. (Each of gph $\mathbf{d}'$ and gph $\mathbf{e}'$ uses all of the points of gph R'.)

First we assign all edges of gph **d** to gph $\mathbf{d}'$, and all edges of gph **e** to gph $\mathbf{e}'$.

For each old point R_d, gph R' has at least one edge connecting R_d to R_{dp}. Assigning one such edge to gph $\mathbf{d}'$ makes gph $\mathbf{d}'$ a connected graph. Assign all remaining edges to gph $\mathbf{e}'$. We claim that gph $\mathbf{e}'$ is also connected.

Let R_{dp} be any new point for which $d \neq 1$. Point R_{dp} is connected to point R_d, in gph R', by more than one edge, by (3). Since gph $\mathbf{d}'$ only used one edge to connect R_d to R_{dp}, these points are also connected in gph $\mathbf{d}'$.

Finally, consider new point R_p. Since $n \neq 1$, n has a prime factor q. Point R_p is connected to point R_{pq} in gph R', by an edge that was not used in gph $\mathbf{d}'$. Therefore point R_p is connected to point R_{pq} in gph $\mathbf{e}'$, and the proof of one theorem is complete. ▮

2.9. THEOREM. *Let P be a maximal ideal of $R = ZG_n$, or more generally, $Z[\zeta_q]\, G_n$ with qn square-free.*

(i) *For every indecomposable, finitely generated R-module M, the localization M_P is either* 0 *or the direct sum of* 1 *or* 2 *indecomposable R_P-modules.*

(ii) *The finitely generated R_P-modules satisfy the Krull–Schmidt theorem.*

Proof. (i) This is true for every Dedekind-like ring: If M is non-artinian, it holds by [L2, 11.8]. If M is artinian, (i) holds because M_P is either zero or $\cong M$. See the proof of [L2, 15.1.]

(ii) Temporarily let R be any Dedekind-like ring, and recall the definitions of $i(k)$ and $j(k)$ from Notation 2.1. It is proved in [L2, 11.10] that the finitely generated R_P-modules satisfy the Krull–Schmidt theorem for every P provided that $i(k) \neq j(k)$ for every k.

In our situation $j(k)/i(k)$ is always a prime number: For $R = ZG_n$ this is proved in Remarks 2.6 [see diagram $\mathscr{P}_{d,dp,h}$]. For $R = Z[\zeta_q]\, G_n$ repeat the reasoning in Remarks 2.6, starting with the more general results in Notation 1.6 and Corollary 1.7. ∎

2.10. *Remarks.* All three possibilities mentioned in Theorem 2.9(i) can actually occur. See the proof of [L2, 11.8].

Localization at primes of Z is quite different from localization at maximal ideals of ZG_n, as in Theorem 2.9. For example, $R = ZG_n$ is always an indecomposable ZG_n-module, by [CR, 32.13]. On the other hand, *for suitable n and suitable primes p dividing n, the p-localization $R_p = Z_p G_n$ is the direct sum of unboundedly many R_p-modules.* To see this, first note that by the proof of [CR, 32.13], the number of idempotent elements of $R_p = Z_p G_n$ is at least as great as the number of prime factors of n/p, hence can be made unboundedly large by suitable choice of n and p. If t is the number of primitive, necessarily orthogonal, idempotents of this commutative noetherian ring, then 2^t is the total number of idempotents of R. Since 2^t can be made unboundedly large, so can t. This proves the claim.

In contrast with Theorem 2.9(ii) we mention that modules over the p-localized group ring $Z_p G_n$ often do not satisfy the Krull–Schmidt theorem. See [CR, 36.3] or [Jo].

3. Projective Dimension 1 (Additive Structure)

In the proofs in this section we assume the reader is familiar with the definition of "simple R-graph" [L2, 11.2], and its relation to indecomposable R-modules [L2, 11.3, statement of 11.4]. As in Remarks 2.6, R_d always denotes $Z[\zeta_d]$.

3.1. Proposition. *Let M be an indecomposable, non-artinian, finitely generated ZG_n-module (n square-free). Then*

(i) *The torsion-free rank [= number of infinite cyclic direct summands] of $(M, +)$ equals that of some indecomposable ideal of ZG_n. In particular, it is $\leqslant n$.*

(ii) *The torsion subgroup of* $(M, +)$ *is n-torsion.* [*Every element is annihilated by some power of n.*]

Proof. By [L2, Theorem 11.4], $M \cong M(\bar{x}; \mathscr{G})$ where $\mathscr{G}$ is simple R-graph built from matrizing choices $\{S_\mu\}$.

Each S_μ is either an ideal of some R_d, in which case its abelian group is torsion-free, or else a module of the form (see [L2, 11.2])

$$(1) \qquad R_{i(k)}/(\ker f_k)^e \qquad \text{or} \qquad R_{j(k)}/(\ker g_k)^e$$

where, in the notation of Remarks 2.6,

$$(2) \qquad f_k = f_{d,dp,h} \qquad \text{and} \qquad g_k = g_{d,dp,h} \qquad \text{for some } (d, dp, h).$$

Thus $\ker f_k$ and $\ker g_k$ are maximal ideals of R_d and R_{dp}, respectively, and each of these ideals contains p. Since p divides n, each of the modules in (1) is n-torsion, hence so is M. Thus (ii) is proved.

To prove (i) define a new graph $\mathscr{H}$ by eliminating almost all torsion from $\mathscr{G}$ as follows. If subgraph $\mathscr{G}_k$ is a single strand beginning and ending with an ideal, as in [L2, 11.2, $(11)_k$], eliminate all of the solid points and let subgraph $\mathscr{H}_k$ be as in (3) below. [Vertex labels are not shown in (3) or (4) when they remain the same in $\mathscr{H}$ as in $\mathscr{G}$.]

$$(3) \qquad \circ \xrightarrow[(k)]{} \circ$$

If $\mathscr{G}_k$ consists of one or two strands with one hollow point each, as in [L2, 11.2, $(12)_k$], eliminate all but of the solid points from each strand, and relabel the solid points(s) as shown as (4), to get $\mathscr{H}_k$:

$$(4) \qquad \circ \xrightarrow[(k)]{} \overset{R_{j(k)}/(\ker g_k)}{\bullet} \qquad\qquad \overset{R_{i(k)}/(\ker f_k)}{\bullet} \xrightarrow[(k)]{\qquad\qquad} \circ$$

Finally, if $\mathscr{G}_k$ is empty, let $\mathscr{H}_k$ be empty. Note that configurations [L2, 11.2, $(13)_k$ and $(14)_k$] cannot occur as $\mathscr{G}_k$ because the graph of ZG_n contains no loops. (See Remarks 2.6 for graph ZG_n.)

Since $\mathscr{G}$ is a connected graph, if one ignores the direction of the arrows, so is $\mathscr{H}$. It is now easy to check that, since $\mathscr{G}$ is a simple R-graph, $\mathscr{H}$ satisfies conditions [L2, 11.2, (5)–(9)], hence $\mathscr{H}$ is a simple R-graph. So $M(\bar{x}; \mathscr{H})$ is indecomposable, by [L2, 11.4]. Since each matrizing choice from which $\mathscr{H}$ is built is an ideal or has length 1, $M(\bar{x}; \mathscr{H})$ is isomorphic to an indecomposable ideal of ZG_n, by [L2, 11.12, (1) and (3)].

Finally, $M(\bar{x}; \mathscr{G})$ and $M(\bar{x}; \mathscr{H})$ have the same torsion-free rank because:

(5) Let $\mathscr{S}$ and $\mathscr{H}$ be simple R-graphs, $R = ZG_n$. Suppose, for each c, that the labels of the points of $\mathscr{S}$ contain an ideal of R_c if and only if the labels of the points of $\mathscr{H}$ do. Then the torsion-free rank of $M(\mathscr{S})$ equals that of $M(\mathscr{H})$.

Let $\mathscr{S}$ be built from matrizing choices $\{S_\mu\}$. The torsion-free rank of $\mathscr{S}$ is clearly $\leqslant$ that of

$$T = \bigoplus \{S_\mu \mid S_\mu \text{ is an ideal of some } R_c\}. \tag{6}$$

So we can prove (5) by showing that $M(\bar{x}; \mathscr{S})$ has a subgroup $\cong T$. Given c, let r_c be any nonzero element of

$$\bigcap_k \{\ker f_k \mid i(k) = c\} \bigcap_k \{\ker g_k \mid j(k) = c\} \tag{7}$$

and let $r = \{r_c\}$. Choose $r_c = 1$ if there are no terms in the intersection (7). If S_μ is an ideal of R_c, then $rS_\mu = r_c S_\mu \subseteq \mathrm{pbk}(\bar{x}; \mathscr{S})$, and

$$T \cong rT \subseteq \mathrm{pbk}(\bar{x}; \mathscr{S}). \tag{8}$$

Since rT has zero intersection with the (artinian) kernel of the map: $\mathrm{pbk}(\bar{x}; \mathscr{S}) \to M(\bar{x}; \mathscr{S})$, (8) shows that $M(\bar{x}; \mathscr{S})$ has a subgroup $\cong T$, as desired. ∎

3.2. THEOREM. *Let n be a square-free integer $\geqslant 2$, and X a finite, nonzero n-torsion group with no direct summands of prime order. Then the additive group $ZG_n \oplus X$ can be given the structure of an* indecomposable, *faithful ZG_n-module of projective dimension* 1. *In fact, this can be done in at least* $|\mathrm{Pic}\, ZG_n|$ *non-isomorphic ways.*

Proof. Let R be the *ring* ZG_n. It is proved in [L2, 14.1] that, if M is a faithful (finitely generated) R-module of projective dimension 1, the number of non-isomorphic R-modules N such that $M_P \cong N_P$ for all maximal ideals P is equal to the order of $\mathrm{Pic}\, R$. Moreover the proof of [L2, 14.1] shows that these modules N all have isomorphic additive groups. Therefore it suffices to construct a single R-module M with the properties described in the theorem.

Let $\mathscr{H} = \mathrm{gph}\, R$. (See Remarks 2.6.) Since $R = ZG_n$ is an indecomposable ring, by [R, 32.13], $\mathscr{H}$ is connected [see 2.2(2)] and is therefore a simple R-graph.

To obtain our desired module M we modify $\mathscr{H}$, getting a new simple R-graph $\mathscr{S}$; then we set $M = M(\bar{1}; \mathscr{S})$. Our modifications will be made in one subgraph $\mathscr{H}_k$ at a time; and we remind the reader that, in the detailed notation of Remarks 2.6, the subscript k becomes a triple of subscripts:

$$k = (d, dp, h). \tag{1}$$

Each subgraph $\mathscr{H}_{d,dp,h}$ is

$$\text{(2)} \qquad \overset{R_d}{\circ} \xrightarrow[(d,dp,h)]{\qquad\qquad} \overset{R_{dp}}{\circ} \quad .$$

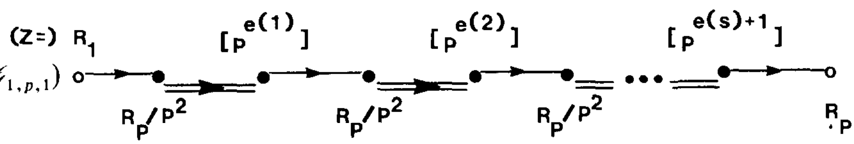
Let $\mathscr{G}_{d,dp,h} = \mathscr{H}_{d,dp,h}$ except when $d = 1$. Coordinate ring $R_1 = Z$ of $\tilde{R}$ has only one maximal ideal containing each prime number p. So every subscript $k = (1, p, h)$ actually has the form $k = (1, p, 1)$.

Now fix a prime divisor p of n. Let the p-primary component of the group X be

$$\text{(3)} \qquad [p^{e(1)}] \oplus [p^{e(2)}] \oplus \cdots \oplus [p^{e(s)}] \qquad [\text{each } e(\mu) \geqslant 2]$$

where $[p^e]$ denotes a cyclic group of order p^e. Each $e(\mu) \geqslant 2$ because of the "no direct summands of prime order" hypothesis.

Let P be *the* maximal ideal of $R_p = Z[\zeta_p]$ that contains p. Finally, let $\mathscr{G}_{1,p,1}$ be obtained from $\mathscr{H}_{1,p,1}$ by inserting additional points and edges as shown:



We have omitted the label $(1, p, 1)$ on each edge. Note that the last summand in (3) has been changed to $[p^{e(s)+1}]$ in $\mathscr{G}_{1,p,1}$. Incidentally, the reason for the "no cyclic direct summands" hypothesis is to make it easy to deal with the requirement that each solid point in $\mathscr{G}_{1,p,1}$ be labeled by a matrizing choice of length $\geqslant 2$. See [L2, 11.2, (6)].

Complete the definition of $\mathscr{G}$ by doing the above insertions for every prime p such that the p-primary component of X is nonzero. Then $\mathscr{G}$ is connected because $\mathscr{H}$ is connected, so $\mathscr{G}$ is a simple R-graph, and therefore $M = M(\bar{1}; \mathscr{G})$ is an indecomposable, non-artinian R-module. (See [L2, 11.4].) M is faithful because the matrizing choices from which it is built contains an ideal of every R_d, namely, R_d itself.

M has projective dimension $\leqslant 1$ because every $\mathscr{G}_k$ is a single strand beginning and ending with a hollow point. See [L2, 12.2 (Graph Version)].

M is not projective because its additive group is not torsion-free. Therefore the projective dimension of M is 1.

Before determining the additive group of M we pause for a lemma.

Given modules P and V over some ring R, we say that P is *projective wrt* ("with respect to") V if, for every R-module homomorphism f' and epimorphism g', as in the right-hand diagram of (3.3), there exists an R-module homomorphism $\varphi\colon P \to V$ such that $f' = g'\varphi$:

$$
\begin{tikzcd}
W \arrow[dr, "f"] & & & P \arrow[dr, "f'"] & \\
& \bar{W} & & & \bar{V}. \\
P \arrow[ur, "g"'] & & & V \arrow[ur, "g'"'] &
\end{tikzcd}
\tag{3.3}
$$

3.3. LEMMA (Change of Basis). *Let homomorphisms of modules over some ring be given, as shown in diagram* (3.3). *Set*

(i) $\operatorname{pbk}(W \oplus P \oplus V) = \{(w, p, v) \in W \oplus P \oplus V \mid f(w) = g(p)$ and $f'(p) = g'(v)\}$.

(ii) $\operatorname{pbk}(W \oplus P) = \{(w, p) \in W \oplus P \mid f(w) = g(p)\}$.

Suppose that P is projective wrt *V. Then*

(iii) $\operatorname{pbk}(W \oplus P \oplus V) \cong \operatorname{pbk}(W \oplus P) \oplus \ker g'$

by an automorphism of $W \oplus P \oplus V$ that equals the identity on $W \oplus 0 \oplus V$.

Proof. Since P is projective wrt V, there exists $\varphi\colon P \to V$ such that $f' = g'\varphi$. Define an automorphism θ of $W \oplus P \oplus V$ by

$$\theta(w, p, v) = (w, p, v - \varphi(p)). \tag{1}$$

Then θ is the identity on $W \oplus V$ as desired. To prove that θ takes the left-hand side of (iii) to the right-hand side, take $(w, p, v) \in \operatorname{pbk}(W \oplus P \oplus V)$. Then $f'(p) = g'(v)$, so

$$g'(v - \varphi(p)) = g'(v) - g'\varphi(p) = f'(p) - f'(p) = 0$$

as desired. For the opposite inclusion, take an element (w, p, k) of the right-hand side of (iii). Then $(w, p, \varphi(p) + k)$ belongs to the left-hand side of (iii) and is taken by θ to (w, p, k). ∎

Completion of proof of Theorem 3.2. We show that the additive group of $M = M(\bar{1}; \mathscr{S})$ is $\cong ZG_n \oplus X$ by repeatedly changing the graph until the additive structure of M becomes clear. Recall the construction of M from $\mathscr{S}$. (See [L2, 11.3] for details.) Let

$$\operatorname{dir}(\mathscr{S}) = \bigoplus_{\mu} S_\mu \tag{4}$$

where $\{S_\mu\}$ is the collection of labels of the points of $\mathscr{S}$, and each S_μ appears in (4) as often as it appears in $\mathscr{S}$. To form M, first form the submodule $\mathrm{pbk}(\bar{1};\mathscr{S}) = \mathrm{pbk}(\mathscr{S})$ of $\mathrm{dir}(\mathscr{S})$ indicated by the pullback edges of $\mathscr{S}$. Then $M = \mathrm{pbk}(\mathscr{S})/\bar{K}(\mathscr{S})$ where $\bar{K}(\mathscr{S})$ accomplishes the socle amalgamations indicated by the amalgamation edges of $\mathscr{S}$.

Fix some p such that $\mathscr{S}_{1,p,1}$ has at least one solid point. Let $V = R_p/P^2$ be the label of the left-most solid point of $\mathscr{S}_{1,p,1}$, *considered as a coordinate module of* $\mathrm{dir}(\mathscr{S})$. Other points of $\mathscr{S}_{1,p,1}$ might be labeled by R_p/P^2 but we will not call their labels V. We prove:

(5) Form graph $\mathscr{S}'$ by making two changes in $\mathscr{S}$: Delete the edge of $\mathscr{S}_{1,p,1}$ that connects R_1 to $V = R_p/P^2$, and replace the label V by P/P^2. Then the R-modules $\mathrm{pbk}(\mathscr{S})/\bar{K}(\mathscr{S})$ and $\mathrm{pbk}(\mathscr{S}')/\bar{K}(\mathscr{S}')$ have isomorphic additive groups.

To prove this, think of $\mathrm{pbk}(\mathscr{S})$ in the form

$$\mathrm{pbk}(\mathscr{S}) = \mathrm{pbk}(W \oplus R_1 \oplus V) \qquad (V = R_p/P^2) \tag{6}$$

as follows. First think of (4) in the form

$$\mathrm{dir}(\mathscr{S}) = (\text{"other terms"}) \oplus R_1 \oplus V. \tag{7}$$

Let W be the pullback of "other terms" determined by the pullback edges of $\mathscr{S}$ that connect them. Then use the remaining pullback edges of $\mathscr{S}$ to form the pullback on the right-hand side of (6). Some of these edges connect "other terms" to R_1, and one connects R_1 to V; but none connect "other terms" to V (see $\mathscr{S}_{1,p,1}$).

Now consider all modules in (6) as Z-modules, rather than R-modules. Then $R_1 = Z$ is projective, hence projective wrt V. By the Change of Basis Lemma, applied to (6), we get a Z-isomorphism

$$\mathrm{pbk}(\mathscr{S}) \cong \mathrm{pbk}(W \oplus R_1) \oplus P/P^2 = \mathrm{pbk}(\mathscr{S}'). \tag{8}$$

To complete the proof of (5) we show that the additive isomorphism in (8) takes $\bar{K}(\mathscr{S})$ onto $\bar{K}(\mathscr{S}')$.

Note that $\mathrm{dir}(\mathscr{S}')$ is formed from (4) by replacing the summand $V = R_p/P^2$ by P/P^2. Since $\mathscr{S}$ and $\mathscr{S}'$ have the same amalgamation edges—except that label P/P^2 replaces V at one point—we see that

$\bar{K}(\mathscr{S}) = \bar{K}(\mathscr{S}') \subseteq \bigoplus \{S_\mu \mid S_\mu \neq R_1\}$ (Note: No amalgamation edges touch point R_1.)

so, in the notation of (6), $\bar{K}(\mathscr{S}) = \bar{K}(\mathscr{S}') \subseteq W \oplus V$. By the Change of Basis Lemma, the isomorphism in (8) is the identity on $W \oplus V$, and this completes the proof of (5).

Now change notation: Suppose changes (5) have been made, and write $\mathscr{G}$ instead of $\mathscr{G}'$. Since P/P^2 is the socle of the R_p-module R_p/P^2, the additive group of P/P^2 has order p. Since this is going to be amalgamated with the subgroup of $[p^{e(1)}]$ that has order p, *we can delete point P/P^2 from $\mathscr{G}_{1,p,1}$*, along with its attached amalgamation edge, and this deletion does not change the isomorphism class of the R-module $\mathrm{pbk}(\mathscr{G})/\bar{K}(\mathscr{G})$. Next we claim:

(9) The pullback edge connecting $[p^{e(1)}]$ to the next R_p/P^2 in $\mathscr{G}_{1,p,1}$ can be deleted, provided we replaced that R_p/P^2 by P/P^2.

The proof of this is similar to that of (5): Let $\mathscr{G}'$ be the altered graph. To see that $\mathrm{pbk}(\mathscr{G})$ is additively isomorphic to $\mathrm{pbk}(\mathscr{G}')$ apply the Change of Basis Lemma to the situation

$$\mathrm{pbk}(0 \oplus [p^{e(1)}] \oplus R_p/P^2) \tag{10}$$

where the pullback homomorphisms connecting the last two summands in (10) are provided by the pullback edge connecting $[p^{e(1)}]$ to R_p/P^2 in $\mathscr{G}_{1,p,1}$. Again we consider all modules merely as Z-modules. The second summand $[p^{e(1)}]$ in (10) is projective wrt the third because—since $e(1) \geqslant (2)$—both are modules over $Z/p^{e(1)}Z$ and $[p^{e(1)}]$ is a *free* module over $Z/p^{e(1)}Z$.

As before, make the changes (9) and then delete P/P^2 and its attached amalgamation edge. Continuing in this way, $\mathscr{G}_{1,p,1}$ eventually takes the form.

$$\begin{array}{cccccc} R_1 & [p^{e(1)}] & [p^{e(2)}] & & [p^{e(s)+1}] & R_p \\ \circ & \bullet & \bullet & \cdots & \bullet \longrightarrow & \circ \end{array} . \tag{11}$$

To remove the last pullback edge from (11), apply the Change of Basis lemma once more, this time to the pullback of $0 \oplus R_p \oplus [p^{e(s)+1}]$ determined by the pullback edge in (11). Since $R_p = Z[\zeta_p]$ is a free Z-module the projectivity hypothesis needed by the lemma is satisfied. The lemma allows us to delete the pullback edge in (11) provided we replace $[p^{e(s)+1}]$ by $[p^{e(s)}]$.

After doing this for every p that divides n and such that $\mathscr{G}_{1,p,1}$ has at least one solid point, we obtain a new graph $\mathscr{H}'$ in which every $[p^e]$ is an isolated point. Let $\mathscr{H}''$ be the graph obtained from $\mathscr{H}'$ by deleting these isolated points. Then

$$M \cong \mathrm{pbk}(\mathscr{H}')/\bar{K}(\mathscr{H}') = \mathrm{pbk}(\mathscr{H}'')/\bar{K}(\mathscr{H}'') \oplus X \tag{12}$$
$$\text{(additive isomorphism).}$$

Note that both denominators in (12) are zero because no amalgamation edges remain. Thus all that remains is to compute the rank of the torsion-free group $\mathrm{pbk}(\mathscr{H}'')$.

Exactly as in the proof of Proposition 3.1(5) we show that inserting or deleting pullback edges into $\mathscr{H}''$ does not change the torsion-free rank of pbk($\mathscr{H}''$). By inserting a pullback edge into each $\mathscr{H}''_{1,p,1}$ that consists of two isolated (hollow) points, we obtain the graph $\mathscr{H} = \text{gph } ZG_n$.

Since pbk($\mathscr{H}$) actually *equals* ZG_n the proof is now complete. ∎

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